Math 3280 Tutorial 11.

Recall:

① $E[X]=E[E[X|Y]]$

② $M_X(t)=E[e^{tX}]$. moment generating function.

③ Two inequalities:

Markovian ineq:

$$P(X\geq a)\leq \frac{E[X]}{a} \qquad a>0.$$

Chebyshev's ineq:

$$P(|X-\mu|\geq \varepsilon)\leq \frac{\sigma^2}{\varepsilon^2}. \qquad \mu=E[X].$$
$$\sigma^2=Var(X).$$

1. If $X_1 \cdots X_n$ are n i.i.d. distributed r.v.s having uniform distribution over $(0,1)$.

$E[\max(X_1,X_2,\dots,X_n)]=\frac{n}{n+1}$. (proved in tutorial 10).

find $E[\min(X_1,X_2,\dots,X_n)]=\frac{1}{n+1}$.

Solution:

$X=\min(X_1,X_2,\dots,X_n)$.

$$\begin{aligned}
P(X>x)&=P(\min(X_1,X_2,\dots,X_n)>x)\\
&=P(X_1>x,X_2>x,\dots,X_n>x)\\
&=P(X_1>x)\cdot P(X_2>x)\cdots P(X_n>x)\\
&=(1-x)\cdot(1-x)\cdots(1-x)\\
&=(1-x)^n.
\end{aligned}$$

$X \in (0,1)$

$$E[X] = \int_0^1 P(X>x)\,dx = \int_0^1 (1-x)^n\,dx$$

$$= -\frac{(1-x)^{n+1}}{n+1}\Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{n+1}.$$

2. Show that

$$Cov(X, E[Y|X]) = Cov(X,Y).$$

Solution:

$$Cov(X, E[Y|X]) = E[X \cdot E[Y|X]] - E[X] \cdot \underbrace{E[E[Y|X]]}_{=E[Y]}$$

$$= E[E[XY|X]] - E[X] \cdot E[Y]$$

$$= E[XY] - E[X] \cdot E[Y]$$

$$= Cov(X,Y).$$

3. Show that

$$E[(X-Y)^2] = E[X^2] - E[Y^2]$$

where $Y = E[X|Z]$.

Solution:

$$E[(X-Y)^2] = E[X^2] - 2\cdot E[XY] + E[Y^2]$$

$$= E[X^2] - 2\cdot \underline{E[X\cdot E[X|Z]]} + E[Y^2].$$

$$= E[X^2] - 2\cdot E[Y^2] + E[Y^2]$$

$$= E[X^2] - E[Y^2].$$

4. Let Z be a standard normal r.v., and for a fixed x, set

$$X = \begin{cases} Z & \text{if } Z > x. \\ 0 & \text{o.w.} \end{cases}$$

Show that

$$E[X] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Solution:

$$E[X] = \underline{E[X\cdot 1_{\{Z>x\}}] + \overset{=0}{E[X\cdot 1_{\{Z\le x\}}]}}.$$

$= E[X \cdot 1_{\{Z>x\}}]$

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$= \int_x^\infty z f_Z(z) dz$

$= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} d(\frac{z^2}{2})$

$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{z=x}^{z=\infty}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

5. Let X be a non-negative r.v. having distribution function F.

Show that

$$E[X^n] = \int_0^\infty n \cdot x^{n-1} \bar{F}(x)\, dx.$$

where

$$\bar{F}(x) = 1 - F(x).$$

Solution:

$$X^n = \int_0^X n \cdot y^{n-1}\, dy$$

$$= \int_0^\infty n \cdot y^{n-1}\, 1_{\{X > y\}}\, dy$$

$$E[X^n] = E[\int_0^\infty n \cdot y^{n-1}\, 1_{\{X > y\}}\, dy]$$

$$= \int_0^\infty E[n \cdot y^{n-1}\, 1_{\{X > y\}}]\, dy$$

$$= \int_0^\infty n \cdot y^{n-1}\, E[1_{\{X > y\}}]\, dy$$

$$= \int_0^\infty n \cdot y^{n-1}\, \bar{F}(y) \cdot dy$$

6. Suppose that $E[X] = Var(X) = 20$.

Prove that

$$\frac{19}{20} \le P(0 < X < 40) \le 1$$

Solution:

$$P(0 < X < 40) = P(-20 < X - 20 < 20)$$

$$= P(|X - 20| < 20)$$

$$= 1 - P(|X - 20| \geq 20)$$

By cheyshov's inequality,

$$P(|X - \underbrace{20}_{= E[X]}| \geq 20) \leq \frac{Var(X)}{20^2} = \frac{1}{20}.$$

$$\geq 1 - \frac{1}{20}$$

$$= \frac{19}{20}. \quad \square$$